DBM20083-DISCRETE MATHEMATICS

Chapter 5

Product Rule → $x^a \cdot x^b = x^{(a+b)}$

ex) $(x^2y^3)(x^4y^2) =$

$(n\ C\ r) = \dfrac{n!}{(n-r)!\,r!}$

$^nP_r = \dfrac{n!}{(n-r)!}$

Chapter 5: BASIC COUNTING RULES

5.1 Derive counting principle

- 5.1.1 Describe the used of counting
- 5.1.2 Describe with examples the following basic decomposition rules/ counting principle
  - 5.1.1a Sum Rule
  - 5.1.2b Product Rule
- 5.1.3 Identify with example the more complex counting problems typically require a combination of the sum and product rules
- 5.1.4 Solve problem using the basic counting principle rule

BASIC COUNTING PRINCIPLE

When there are **m** ways to do one thing, and **n** ways to do another, BUT we can't do both at the same time, then there are **m + n** ways to choose one of the actions.

(OR CONCEPT)

Example:
A college library has 40 textbooks on sociology and 50 textbooks on anthropology. To learn about sociology or anthropology, a student can choose from 40 + 50 = 90 textbooks

Example:
A student can choose a math project from one of two lists. The two lists contain 17 and 23 possible projects, respectively, and no project is on both lists. That means the student can choose 17 + 23 = 40 possible projects.

BASIC COUNTING PRINCIPLE

When there are **m** ways to do one thing, and n ways to do another, then there are **m × n** ways of doing **both**.

(AND CONCEPT)

Example:

You have 3 shirts and 4 pants. That means 3×4=12 different outfits.

Example:

There are 6 flavors of ice-cream, and 3 different cones. That means 6×3=18 different single-scoop ice-creams you could order.

Example

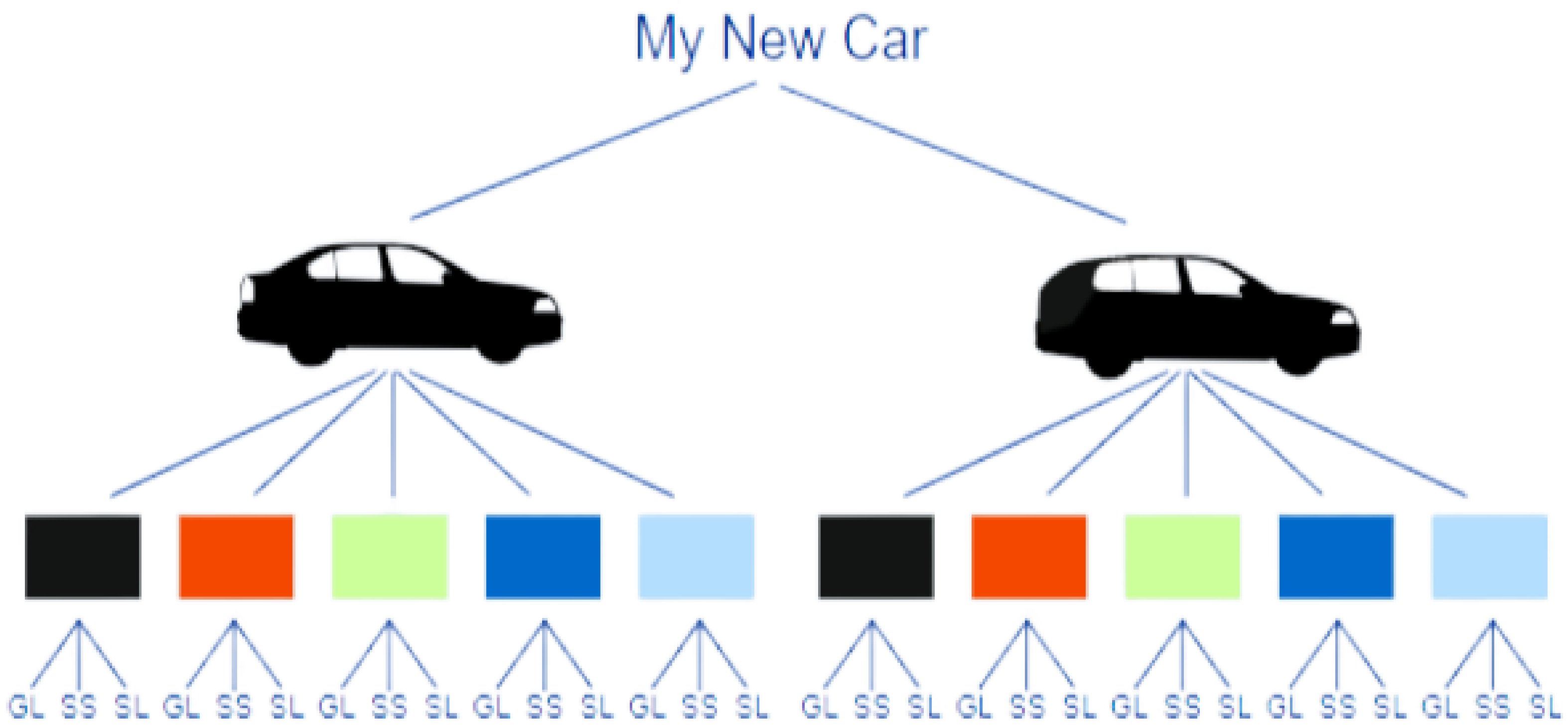
You are buying a new car. There are 2 body styles:

There are 5 colors available:

There are 3 models: GL(standard model), SS(sports model with bigger engine) and SL(luxury model with leather seats). How many total choices?

Solution

You can see in this "tree" diagram:



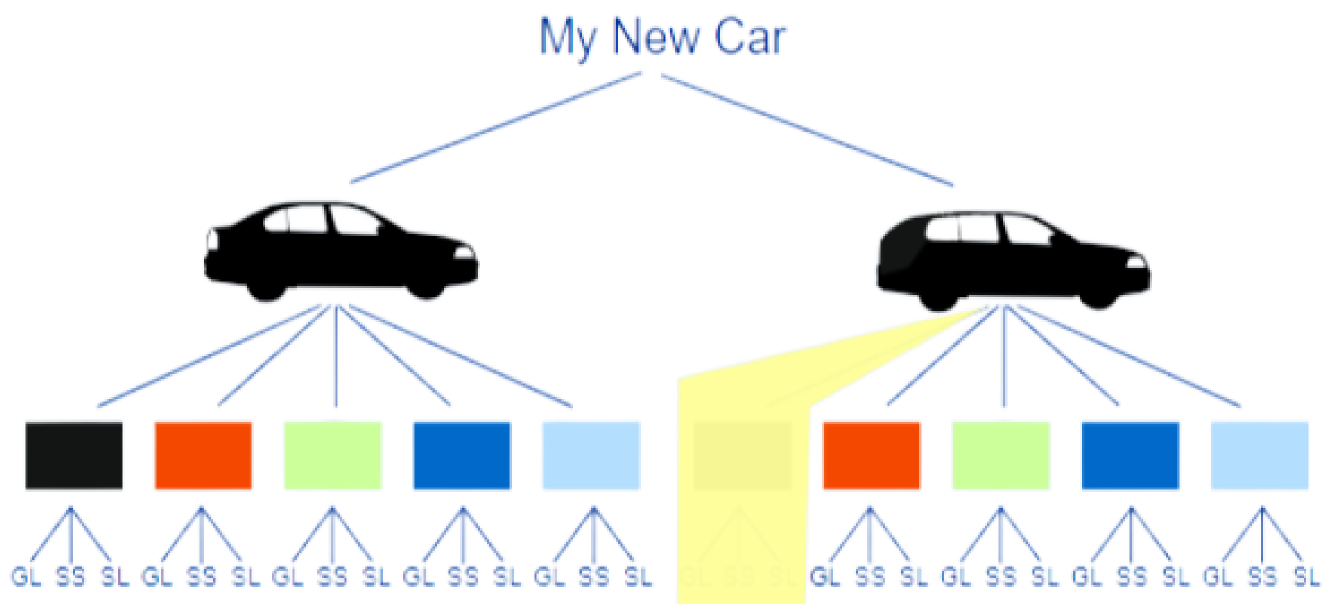
You can count the choices, or just do the simple calculation:

Total Choices = $2 \times 5 \times 3 = 30$

SOLUTION

You are buying a new car ... but ... the salesman says "You can't choose black for the hatchback" ... well then things change!

You can see in this "tree" diagram:



You now have You can count the choices, or just do the simple calculation. But you can still make your life easier with this calculation:

Choices = 5×3 + 4×3 = 15 + 12 = 27

Exercise A

1. How many ways of a student can choose a calculus professor if there are 8 male professors and 5 female professors who teach calculus class?
2. How many outfits can be made from 4 pairs of pants, 3 shirts, and 2 pairs of shoes?
3. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?
4. Make a ***tree diagram*** to answer this one. How many ways can you arrange a fun evening out if you have 3 choices for restaurants, 3 choices for movies, and 2 choices for a friend to take along? You choose the names of the movies, restaurants, and friends.

Exercise A

5. How many ways can you arrange 4 books on the same shelf? You can use a single letter to represent a book title (such as A, B, C, and D). (Hint: there are 4 to choose from for the first position, leaving 3 to choose from for the second position, etc.)
6. A restaurant offers a meal set where there are 5 choices of appetizer, 10 choices of main meal and 4 choices of dessert. How many different possible meals does the restaurant offer?

Exercise A

7. Sarah goes to her local pizza parlor and orders a pizza. She can choose either a large or a medium pizza, can choose one of seven different toppings, and can have three different choices of crust. How many different pizzas could Sarah order?
8. Derek must choose a four-digit PIN number. Each digit can be chosen from 0 to 9. How many different possible PIN numbers can Derek choose?

Exercise A

9. For her literature course, Rachel has to choose one novel to study from a list of four, one poem from a list of six and one short story from a list of five. How many different choices does Rachel have?
10. Jenny has nine different skirts, seven different tops, ten different pairs of shoes, two different necklaces and five different bracelets. In how many ways can Jenny dress up?

COMBINATION OF SUM AND PRODUCT RULE

Example:

Calvin wants to go to Milwaukee. He can choose from **3 bus services** or **2 train services** to head from home to downtown Chicago. From there, he can choose from **2 bus services** or **3 train services** to head to Milwaukee. How many ways are there for Calvin to get to Milwaukee?

Solution:

- He has 3 + 2 = 5 ways to get to downtown Chicago. **(Rule of sum)**
- From there, he has 2 + 3 = 5 ways to get to Milwaukee. **(Rule of sum)**
- Hence, he has 5 x 5 = 25 ways to get to Milwaukee in total.
- **(Rule of product)**

COMBINATION OF SUM AND PRODUCT RULE

Example:

Pastry shop menu:

6 kinds of muffins, 8 kinds of sandwiches, hot coffee, hot tea, ice tea, cola, orange juice

Buy either a muffin and a hot beverage, or a sandwich and a cold beverage.

How many possible purchases?

Solution:

- Muffin and hot beverage=6 . 2 = 12 (Product Rule)
- Sandwich and cold beverage= 8 . 3 = 24 (Product Rule)
- Hence, there are 12 + 24 = 36 ways to purchase. (Sum Rule)

Exercise B

1. A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?
2. A restaurant offers 5 choices of appetizer, 10 choices of the main course and 4 choices of dessert. A customer can choose to eat just one course, or two different courses, or all three courses. Assuming that all food choices are available, how many different possible meals does the restaurant offer? (NOTE: When you eat a course, you only pick one of the choices).

Exercise B

3. Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Chapter 5: BASIC COUNTING RULES

5.2 Compute permutations and combinations

- 5.2.1 Define permutation
- 5.2.2 Describe permutations with and without repetition
- 5.2.3 Solve counting problems by using permutations
- 5.2.4 Describe combinations with and without repetition
- 5.2.5 Solve counting problems by using combinations

PERMUTATION

Definition: An arrangement or listing in which the order or placement is important.

Two types of permutation:

1. REPETITION is ALLOWED

Example; safety pin.

It could be "333".

2. REPETITION is NOT ALLOWED

Example: the first three people in a running race. You can't be first *and* second.

To help you to remember, think "Permutation ... Position"

1. REPETITION IS ALLOWED

When a thing has ndifferent types ... we have nchoices each time!

For example: choosing 3of those things, the permutations are:

$n \times n \times n$ (n multiplied 3 times)

More generally: choosing rof something that has ndifferent types, the permutations are:

$n \times n \times \ldots$ (r times)

(In other words, there are n possibilities for the first choice, THEN there are n possibilities for the second choice, and so on, multiplying each time.)

Which is easier to write down using an exponent of r:

$n \times n \times \ldots$ (r times) $= n^r$

So, the formula is simply:

$$\boldsymbol{n}^{r}$$

Where n is the number of things to choose from, and we choose r of them, repetition is allowed and order matters.

Example

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Hence, by the product rule, there are a total of $2^7 = 128$ different bit strings of length seven.

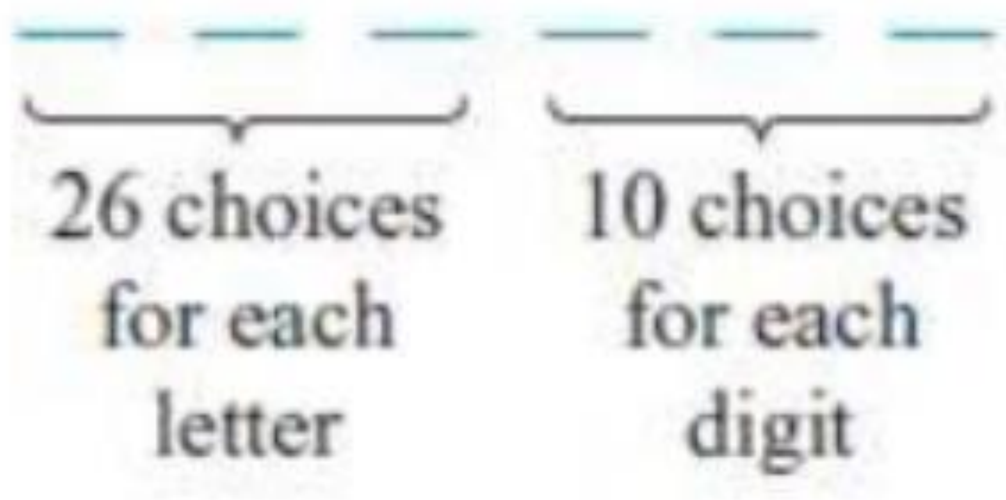
EXAMPLE

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits? (Note: Repetition of English letters and digits are allowed)

<u>Solution:</u>

There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits.

Hence, by the product rule, there are a total of 26.26.26.10.10.10 = 17,576,000 possible license plates.



2. REPETITION IS NOT ALLOWED

The number of permutations of n different items, taken r at a time is

$$^{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!}$$

<u>Example:</u>

There are 16 balls tagged with number 1 till 16. How many ways can we pick 3 balls without repeating the same balls.

<u>Solution:</u>

So, your first choice would be 16, the second choice is 15 and the third is 14 possibilities. **16**. **15**. **14** = **3360** ways OR

$$^{16}P_{3} = P(16,3) = \frac{16!}{(16-3)!} = \frac{16!}{(13)!} = 3{,}360 \text{ ways}$$

EXAMPLE

There are 16 balls tagged with number 1 till 16, how many ways can we pick **all balls without repeating the same balls.**

<u>Solution:</u>

Your first choice would be 16, the second choice is 15, the third is 14 possibilities and etc.

$$16 \times 15 \times 14 \times \cdots = 20,922,789,888,000 \text{ ways}$$

Or

$${}^{16}P_{16} = 16! = \underline{20,922,789,888,000} \text{ ways}$$

Example

A class has 10 students: A, B, C, D, …, I, J. 4 students are to be seated in a row for a picture:

BCEF, CEFI, ABCF, …

How many such arrangements?

Solution:

Filling for a position: A stage of the counting procedure (Product Rule)

$$10 \times 9 \times 8 \times 7 = 5040$$

Example

a) Find the number of arrangements for the word BOBBY

b) Find the number of arrangements for the word BENZENE

Solution:

a) BOBBY have repetition letter which is B, and it's repeated 3 times.

$$P(5;3) = \frac{5!}{3!} = \frac{120}{6} = 20$$

b) BENZENE has repetition letters which are E & N. E is repeated 3 times and N is repeated 2 times.

$$P(7;3,2) = \frac{7!}{3!2!} = 420$$

EXAMPLE

A certain password consists of 6 digits. Find the number of possible ways this password can be formed

a) If **all the digits can be repeated**

b) If **no digit can be repeated**

c) If the password **cannot begin with the digit 0 and the digits can be repeated**

Solution:

There are all together 10 digits from 0 to 9.

a) 10^6 = 1,000,000 ways

b) $^{10}P_6 = \frac{10!}{(10-6)!} = 151{,}200$

c) 9×10×10×10×10×10 = 900,000 ways

IMPORTANT

Without repetition our choices get reduced each time.

The **factorial function** (symbol: !) just means to multiply a series of descending natural numbers. Examples:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$$

$$1! = 1$$

Exercise A

1. How many permutations of 3 different digits are there, chosen from the ten digits 0 to 9 inclusive?
2. A password consists of four different letters of the 26 alphabet. How many different possible passwords are there?
3. A password consists of two letters of the alphabet followed by three digits chosen from 0 to 9. Repeats are allowed. How many different possible passwords are there?
4. Assuming that any arrangement of six letters forms a 'word', how many 'words' of any length can be formed from the letters of the word SQUARE? No repeating of letters.
5. Find the number of ways that a party of seven persons can arrange themselves in a row of seven chairs.

Exercise C

6. Find the number of permutations that can be formed from all the letters of each word.
   a) RADAR
   b) UNUSUAL
7. In how many ways can four mathematics books, three history books, three chemistry books, and two sociology books be arranged on a shelf so that all books of the same subject are together?
8. How many vehicle plate numbers can be made if each plate contains two different letters followed by three different digits?

Exercise C

9. Find the number of permutations that can be formed from the letters of the word ELEVEN
10. How many of the word ELEVEN begin and end with letter E

COMBINATION

A combination focuses on the selection of objects without regard to the order in which they are selected.

Two types of combination:

1. REPETITION is NOT ALLOWED

2. REPETITION is ALLOWED

Example:coins in your pocket

(5,5,5,10,10)

To help you to remember, think ABC and BCA represent equivalent combinations of the letters ABC.

1. REPETITION IS NOT ALLOWED

In general, the number of combinations of n things, taken r at a time is

$$^{n}C_{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

Where n is the number of things to choose from, and we choose r of them

Example

There are 16 **balls** tagged with number 1 till 16. How many ways can we pick 3 combination **balls** without repeating the same **balls**?

Solution:

$$^{n}C_{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

$$^{16}C_{3} = \frac{16!}{3!(16-3)!} = \frac{16!}{3!(13)!} = 560 \text{ combinations}$$

e.g: 123 (a combination), 345, 678, 124, 125, 126, ...,

2. REPETITION IS ALLOWED

In general, the number of combinations of n things, taken r at a time is

$$C(n+r-1, r) = C(n+r-1, n-1) = \frac{(n+r-1)!}{r!\ (n-1)!}$$

Where n is the number of things to choose from, and we choose r of them

EXAMPLE

There are five flavors of ice cream: banana, chocolate, lemon, strawberry and vanilla. You can have three scoops. How many variations will there be?

Solution: You can have 3 scoops and repetition is allowed, so it may be CCC, CCB, CCL ...etc.

By using formula;

$$C(5+3-1, 5-1) = \frac{(5+3-1)!}{3!\ (5-1)!}$$

$$= \frac{7!}{3!4!}$$

$$= 35 \text{ variations}$$

Example

A committee is to be formed 8 men and 4 women. Find the number of ways this committee can be formed consisting of

a) 7 members
b) 5 men and 2 women
c) 7 members with the men more than the women

Solution

a) $^{12}C_7 = C(12, 7) = \dfrac{12!}{7!(12-7)!} = 792$

b) $^{8}C_5 \times {}^{4}C_2 = 336$

c) 7 men : $^{8}C_7 = 8$

6 men : $^{8}C_6 \times {}^{4}C_1 = 112$

5 men : $^{8}C_5 \times {}^{4}C_2 = 336$

4 men : $^{8}C_4 \times {}^{4}C_3 = 280$

There are **4 possibilities** with 7 committee members. Hence, we can add all values.

Number of ways = 8 + 112 + 336 + 280 = 736

EXAMPLE

There are 6 men and 5 women in a room. In how many ways we can choose 3 men
and 2 women from the room?

<u>Solution:</u>

- The number of ways to choose 3 men from 6 men is ${}^6C_3$
- The number of ways to choose 2 women from 5 women is ${}^5C_2$
- Hence, the total number of ways is ${}^6C_3 \times {}^5C_2 = 20 \times 10 = 200$

Example

How many ways can you choose 3 distinct groups of 3 students from total 9 students?

<u>Solution:</u>

Let us number the groups as 1, 2 and 3.

- For choosing 3 students for 1st group, the number of ways - $^9C_3$
- The number of ways for choosing 3 students for 2nd group after choosing 1st group - $^6C_3$
- The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group – $^3C_3$
- Hence, the total number of ways – $^9C_3 \times {}^6C_3 \times {}^3C_3 = 84 \times 20 \times 1 = 1680$

Example

How many different committees of 5 people can be chosen from 10 people?

<u>Solution:</u>

In choosing a committee, order doesn't matter; so we need the number of combinations of 5 people chosen from 10

$= {}^{10}C_5$

$=10!/(5!)(5!)$

$= (10 \times 9 \times 8 \times 7 \times 6)/(5 \times 4 \times 3 \times 2 \times 1)$

$= 30\,240/120$

$= 252$

Example

Jones is the Chairman of a committee. In how many ways can a committee of 5 be chosen from 10 people given that Jones must be one of them?

Solution:

Jones is already chosen, so we need to choose another 4 from 9. In choosing a committee, order doesn't matter; so we need the number of combinations of 4 people chosen from 9

$= {}^9C_4$

$= 9!/(4!)(5!)$

$= (9 \times 8 \times 7 \times 6)/(4 \times 3 \times 2 \times 1)$

$= 3\,024/24$

$= 126$

Exercise C

1. A bag contains six white marbles and five red marbles. Find the number of ways four marbles can be drawn from the bag if
   a) They can be any color
   b) Two must be white and two red
   c) They must all be in the same color

2. John has 8 friends. In how many ways can he invite one or more of them to dinner?

Exercise D

3. In how many ways can a cricket-eleven be chosen out of 15 players? if
   a) A particular player is always chosen
   b) A particular player is never chosen.

4. How many committees of five with a given chairperson can be selected from 12 persons?

SUPPLEMENTARY PROBLEMS

1. How many different signals can be made by 5 flags from 8-flags of different colors?

2. How many words can be made by using the letters of the word "SIMPLETON" taken all at a time?

3. A child has 3 pocket and 4 coins. In how many ways can he put the coins in his pocket?

4. In how many ways can the letters of the word "University" be arranged?

SUPPLEMENTARY PROBLEMS

5. A student is to answer 10 out of 13 questions on an exam
   a) How many choices he has?
   b) How many if he must answer the first two questions?
   c) How many if he must answer the first or second question but not both?
   d) How many if he must answer exactly three out of the first five questions?
   e) How many if he must answer at least three of the first five questions?

Extra Exercise

1. A bag contains 9 discs numbered 1, 2, 3, 4, 5, 6, 7, 8, 9.
   i. Amy chooses 5 discs at random, without replacement, and places them in a row.
      a) How many different 5-digit numbers can be made?
      b) How many different ODD 5-digit numbers can be made?
   ii. Amy's 5 discs are put back in the bag. Mary chooses 5 discs at random, without replacement. Give your answers as EXACT values, find the probability that
      a) The 5 digits include at least 4 odd digits
      b) The 5 digits add up to 33.

Extra Exercise

2. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

3. In how many different ways can the letters of the word 'LEADING' be arranged in such a way that the vowels always come together?

4. In how many different ways can the letters of the word 'CORPORATION' be arranged so that the vowels always come together?

Extra Exercise

5. In how many ways can the letters of the word 'LEADER' be arranged?
6. How many ways the letters of the word ‘ARMOUR’ can be arranged?
7. How many ways the letters of the word ‘CLANKING’ can be arranged?
8. In how many ways, a group of 3 boys and 2 girls can be formed out of a total of 4 boys and 4 girls?
9. How many even numbers of four digits can be formed with the digits 0, 1, 2, 3, 4, 5, 6 and 7; no digit being used more than once?

Extra Exercise

10. How many numbers of four digits greater than 2,400 can be formed with digits 0, 1, 2, 3, 4, 5 & 6; no digit being repeated in any number?
11. How many different words can be formed with the letters of the word ‘FAMILY’ when vowels occupy even places?
12. In how many ways can 5 boys and 4 girls be seated in a row, so that they alternate?
13. In a group, there are 8 women and 7 men, how many groups of 5 women and 5 men can be formed?
14. 8 bottles are to be selected from a bundle of 9 plastic and 7 steel. In how many ways will the bottles with at most 3 steel and at least 4 plastic be selected?

Extra Exercise

15. How many possible two-digit number can be formed by using the digits 3,5,7 (repetition is allowed)?
16. In how many different ways can the letters of the words 'GEPGRAPHY' be arranged such that the vowel must always come together?
17. In how many ways can 5 boys and 5 girls can be seated in a row so that boys and girls are placed alternately?
18. How many 3-digits odd number can be formed from the digits 5,6,7,8,9 if the digits can be repeated.
19. In how many different ways can the letters of the word 'FIGHT' be arranged?

Extra Exercise

20. In how many different ways can the letters of the word 'FIGHT' be arranged?
21. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?
22. How many four-digit numbers can be formed with digits 2,5,6,7, and 8? (repeating digits are not allowed)
23. In how many ways can we sort the letters of the word management so that the comparative position of vowels and consonants remains the same as in MANAGEMENT.

Extra Exercise

24. How many different 6-digit numbers can be formed from the digit 4,5,2,1,8,9?
25. Out of 8 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
26. Find the number of ways that the letters in SAVEMYMATHS may be rearranged if:
    i. no restrictions apply
    ii. there must be an S at each of the arrangement
    iii. the two A's must be together.