DBM20153-DISCRETE MATHEMATICS

Chapter 1

Chapter 1: SETS, RELATIONS AND FUNCTIONS

1.1 Explain sets

1.1.1 Use set notation and operation on sets:

a)Union

b)Intersection

c)Disjoint

d)Difference

e)Complement

f)Symmetric Difference

Chapter 1: SETS, RELATIONS AND FUNCTIONS

1.2 Explain relations

- 1.2.1 Explain properties of relations
- 1.2.2 Explain equivalence relation:
  - a) Reflexive
  - b) Symmetric
  - c)Transitive

BASIC TERMINOLOGY OF SETS AND STANDARD NOTATION IN SETS

- A set is an **unordered collection of objects.**
- The *objectin* a set are called the elements or ***members*** of the set.
- Capital letters like $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, …… are used to denote sets and lowercase letters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}$, $\boldsymbol{y}$, …… are used to denote elements of sets.

- The statement "**p is an element of A**"or equivalently "**p belongs to A**" is written $\boldsymbol{p} \in \boldsymbol{A}$.
- • The statement "**p is not an element of A**" that is the negation of $\boldsymbol{p} \in \boldsymbol{A}$ is written $\boldsymbol{p} \notin \boldsymbol{A}$.

BASIC TERMINOLOGY OF SETS AND STANDARD NOTATION IN SETS

EQUAL SETS

- If and only if their number of elements and the member of elements are exactly same.
- The order of elements and the repetition of elements does not have any relevance here.

Example:

A = {5,6,7},

B = {7,5,6},

C = {5,5,6,6,7,7}

Here, all the three sets, set A, set B and set C are equal.

SPECIAL SYMBOLS FOR SETS

N = the set of **positive integers** : 1, 2, 3, ….

Z = the set of **integers** : ….., -2, -1, 0, 1, 2, …..

Q = the set of **rational numbers**

R = the set of **real numbers**

C = the set of **complex numbers**

EXAMPLE

List the members of these sets.

a) $\{x \mid x \text{ is real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

e) $\{x: x \in N, x \text{ is odd number}, x < 15\}$, where $N = \{1, 2, 3 \ldots\}$

Solution:

a) $\{-1, 1\}$

b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

c) $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

d) $\emptyset$ (empty set which means no element)

e) $\{1, 3, 5, 7, 9, 11, 13\}$

Example

Determine whether each of these pairs of sets are equal.

a) {1, 3, 3, 3, 5, 5, 5, 5, 5} , {5, 3, 1 }

b) {{1} , {1, {1}}

Solution:

a) Yes

b) No

EXERCISE A

1.List the elements of the following sets; here $N = \{1, 2, 3 \ldots\}$.

a) $A = \{ x : x \in N, 3 < x < 10 \}$

b) $B = \{x : x \in N, x \; is \; even$ number, $x < 15 \}$

c) $C = \{x : x \in N, 4 + x = 7\}$

d) $D = \{x : x \in N, x \; is \; odd \; number, x < 20 \}$

e) $E = \{x : x \in N, x$ is a factor of 20, $x \leq 20 \}$

2. Given the universal set $U = \{x : 15 \leq x \leq 26, x \; is \; an \; integer\}$,

$M = \{x : x \; is \; a \; prime \; number\}$;

$N = \{x : x \; is \; a \; multiple \; of \; 3\}$;

$O = \{x : x \; is \; an \; even \; number\}$

a) List the elements of set M, N and O.

b) List the elements of set N' and O' .

c) State the value of $n(N')$ and $n(O')$.

EXERCISE A CONT…

3. List the element of the following sets:

a) $\{x: x \in N, 5 < x < 12\}$, where $N = \{1, 2, 3…\}$

b) $\{x: x \in N$, x is even, $x < 15\}$, where $N = \{1, 2, 3…\}$

c) $\{x: x \in N$, $10 < x< 35$, x with sum of digits less than 6} where $N = \{1, 2, 3…\}$

4. Given that the universal set .

$$\xi = \{x : x \text{ is an integer, } 1 \leq x \leq 10\}$$

$$P = \{x : x \text{ is an odd number}\}$$

$$Q = \{x : x \text{ is a factor of } 12\}$$

a) List the element of set *P* and set *Q*

b) State the value of n(*P*) and n(*Q*)

EXERCISE A CONT...

5. List all the elements of the following sets:

a) A= {x: x is a letter in the word MATHEMATICS}
b) B= {x: x is a month of a year not having 31 days}
c) C= {x: x is a letter before e in the English alphabet}
d) D= {x: x is a vowel in the word “EQUATION”}

6. Determine whether each pair of sets is equal or not

a) {1, 2, 2, 4} , {1, 2, 4}
b) {1, 1, 3} , {3, 3, 1}
c) $\{x \mid x^2 + x = 2\}$, {1, -1}
d) $\{x \mid x \in R \text{ and } 0 < x \leq 2\}$, {1, 2}

VENN DIAGRAM,UNIVERSAL SET,EMPTY SET & SUBSETS

- The Universal Set is the set which contains all the element or objects of other sets, including its own elements.
- It is often shown using the symbol **U or** ξ
- In the Venn Diagram, the Universal Sets is indicated by a rectangle around the sets.

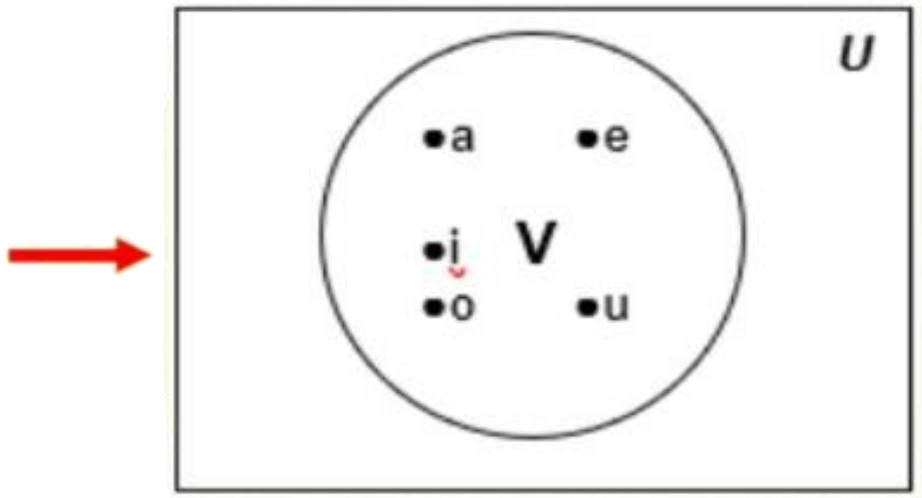
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.

VENN DIAGRAM,UNIVERSAL SET,EMPTY SET & SUBSETS

Example of Venn diagram:

Example 1

The set of vowels, V in the English alphabet.



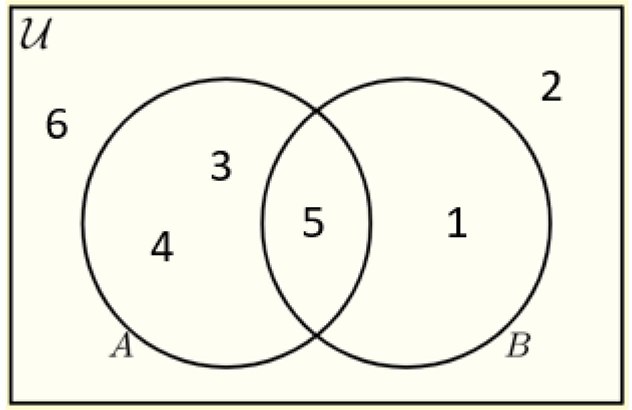
Example 2

Given:

$\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$

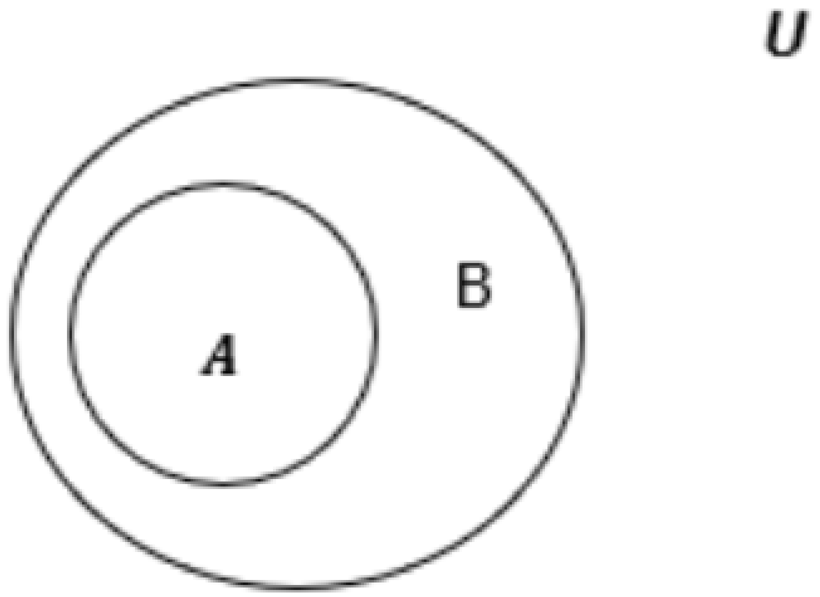
$A = \{3, 4, 5\}$

$B = \{1, 5\}$



VENN DIAGRAM,UNIVERSAL SET,EMPTY SET & SUBSETS

- The set $\boldsymbol{A}$ is said to be a ***subset*** of $\boldsymbol{B}$ if and only if every element of $\boldsymbol{A}$ is also an element of $\boldsymbol{B}$. We use the notation $\boldsymbol{A} \subseteq \boldsymbol{B}$ or $\boldsymbol{B} \supseteq \boldsymbol{A}$ to indicate that $\boldsymbol{A}$ is a subset of the set $\boldsymbol{B}$.



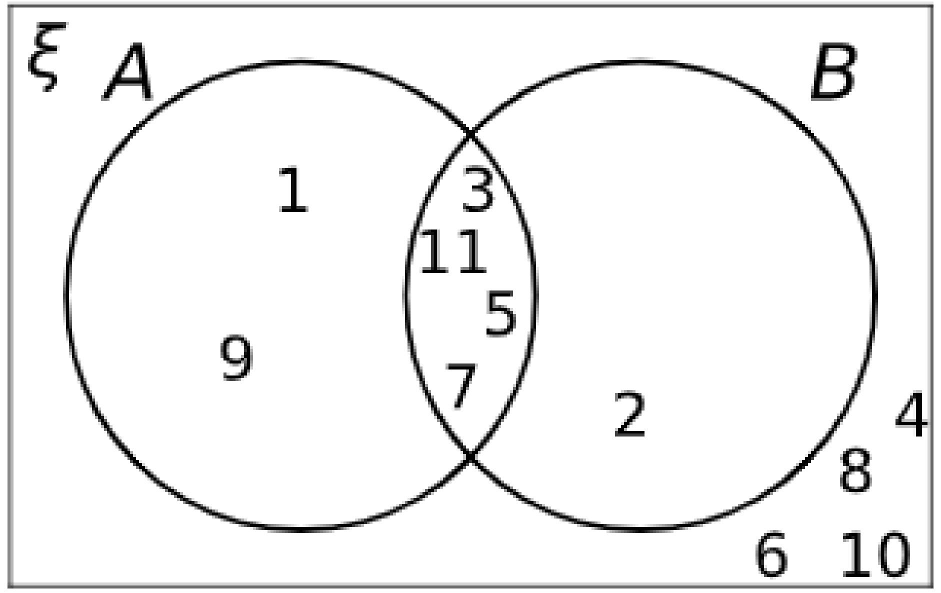
- If $\boldsymbol{A}$ is not a subset of $\boldsymbol{B}$, if at least one element of $\boldsymbol{A}$ does not belong to $\boldsymbol{B}$, we write $\boldsymbol{A} \nsubseteq \boldsymbol{B}$ or $\boldsymbol{B} \nsupseteq \boldsymbol{A}$.
- Empty set is a special set that has no elements. Notation: $\boldsymbol{\emptyset}$ or $\{\ \}$.

EXAMPLE

(a) Suppose that $\xi = \{1,2, 3, 4,5,6,7,8,9,10,11\}$, Put the odd number in A and prime number in B . Draw a Venn diagram for this information.

<u>Solution:</u>

$$A = \{1,3,5,7,9,11\}, \qquad B = \{2,3,5,7,11\}$$



EXAMPLE

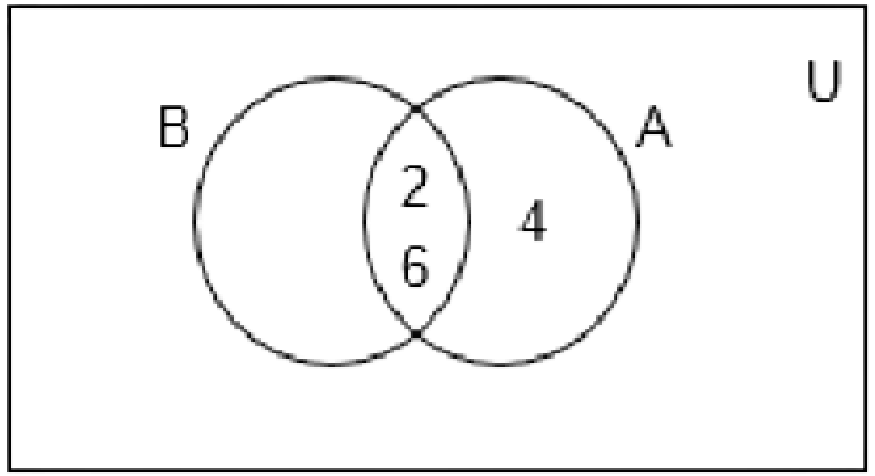
(a) **Suppose that** $A = 2, 4, 6$, $B = 2, 6$, $C = 4, 6$ and $D = 4, 6, 8$. **Determine which of these sets are subsets of which other of these sets**.

(b) **Draw a Venn diagram to illustrate the relationship** $B \subseteq A$ **by using the question above**.

<u>Solution:</u>

a) B ⊆ A; C ⊆ D

b)



EXAMPLE

Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $x \in \{x\}$

d) $\{x\} \subseteq \{x\}$

<u>Solution:</u>

a) False

b) False

c) True

d) True

Exercise B

1. Consider the following sets :

$$\emptyset\ ,\ A = \{1\},\ \ B = \{1, 3\}, C = \{1, 5, 9\},$$

$$D = \{1, 2, 3, 4, 5\}, E = \{1, 3, 5, 7, 9\},$$

$$U = \{1, 2, \ldots\ldots, 8, 9\}$$

Insert the correct symbol $\subseteq$ or $\nsubseteq$ between each pair of set.

a) $\emptyset$, A b) A , B c) B , C d) B , E

e) C , D f) C , E g) D , E h) D , U

2. Use a Venn Diagram to illustrate the relationship

i. $A \subseteq B$ and $B \subseteq C$

ii. $A \subseteq D$ and D$\subseteq$ U

Exercise B cont...

3. The universal set,

$$U = \{x: 10 < x < 35, x \text{ is an integer}\},$$

$$F = \{x: x \text{ is a prime number}\},$$

$$G = \{x: x \text{ is a multiple of } 3\}$$

$$H = \{x: x < 20\}.$$

a) List all the elements of set F, G and H.

b) Draw the Venn diagram to represent all elements of $\mathbf{F} \cup \mathbf{G} \cup \mathbf{H}$ in the universal set.

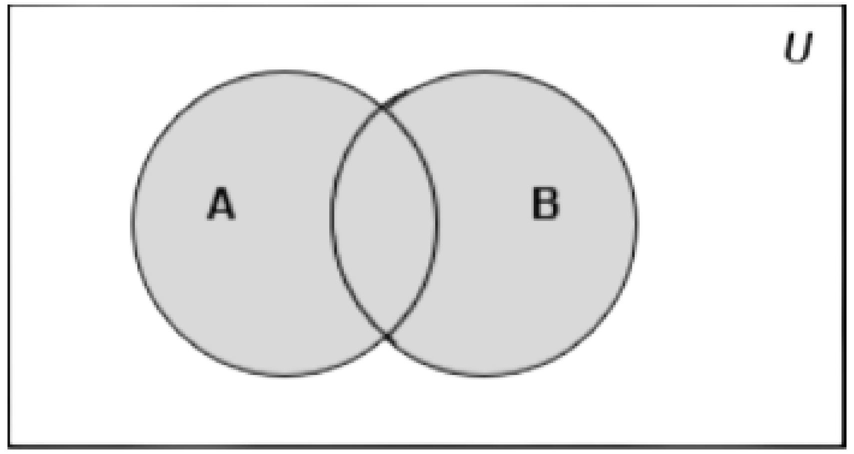
c) Find $\mathbf{n(F \cap H)}$

OPERATION ON SETS

UNION OF THE SETS A AND B

- Denoted by $A \cup B$
- Definition: the set that contains those elements that are either in A or in B, or in both.

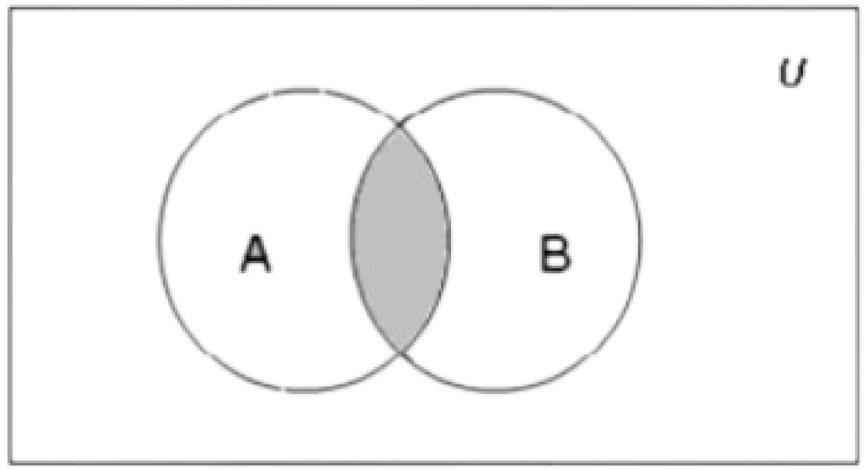
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



INTERSECTION OF THE SETS A AND B

- Denoted by $A \cap B$
- Definition: containing those elements in both A and B.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



OPERATION ON SETS

DISJOINT OF TWO SETS

- Two sets are called **disjoint** if their intersection is the empty set.
- Example:

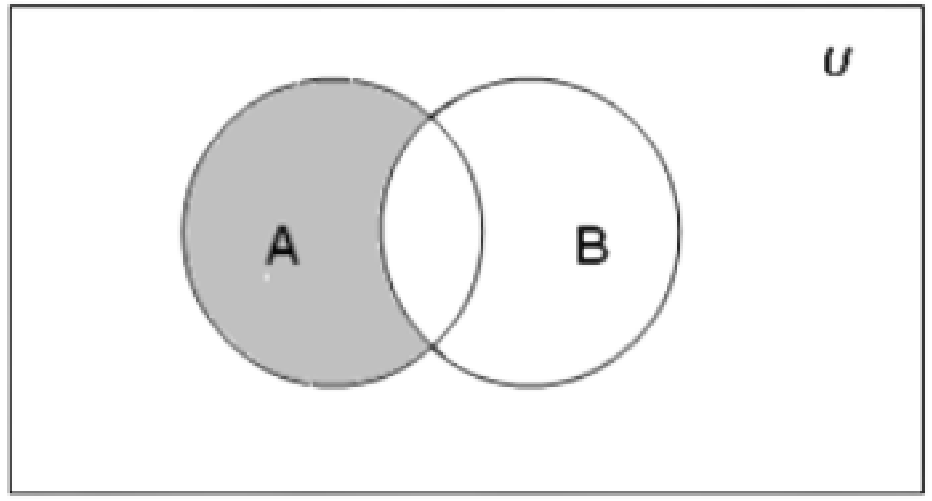
$\boldsymbol{A} = \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7}. \mathbf{9}\}$ and $\boldsymbol{B} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{10}\}$. Because $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are disjoint.

OPERATION ON SETS

DIFFERENCE OF A AND B

- Denoted by $\boldsymbol{A} - \boldsymbol{B}$ or $\boldsymbol{A} \backslash \boldsymbol{B}$
- Definition: the set containing those elements that are in $\boldsymbol{A}$ but not in $\boldsymbol{B}$.

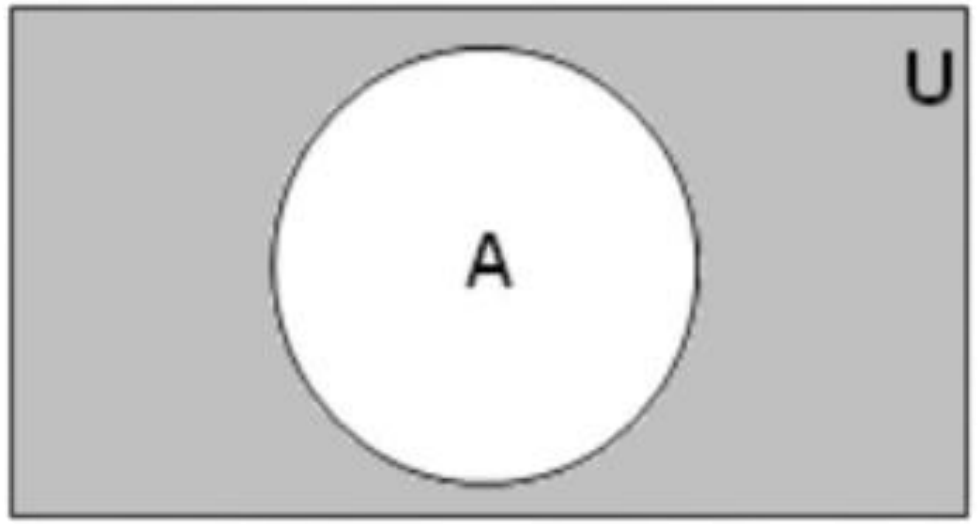
$$A - B \text{ or } A \backslash B = \{x : x \in A, x \notin B\}$$



COMPLEMENT OF SET A

- Denoted by $\boldsymbol{A^c}$ or $\bar{\boldsymbol{A}}$ or $\boldsymbol{A'}$.
- Definition: the ***complement*** of $\boldsymbol{A}$ with respect to $\boldsymbol{U}$ (universal set).

$$A^c = \{x : x \in U, x \notin A\}$$

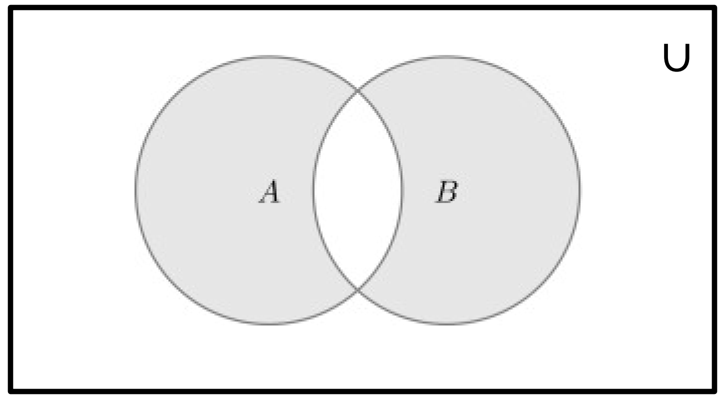


OPERATION ON SETS

SYMMETRIC DIFFERENCE

- The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.
- Example:

  $\boldsymbol{A}$ = {**1**, **3**, **5**, **7**,**9**} and $\boldsymbol{B}$ = {**3, 4, 5, 6,7**}. Hence $\boldsymbol{A} \oplus \boldsymbol{B}$ are {**1,4,6,9**}.



SET IDENTITIES

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

EXAMPLE

1. Let $A = 1, 2, 3, 4, 5$ and $B = 0, 3, 6$. Find:
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B \setminus A$
   e) $A \oplus B$

2. Draw the Venn Diagram for each of these combinations of sets A and B.
   a) $A \cap B^{\boldsymbol{c}}$
   b) $(B \setminus A)^{\boldsymbol{c}}$

Solution

1.

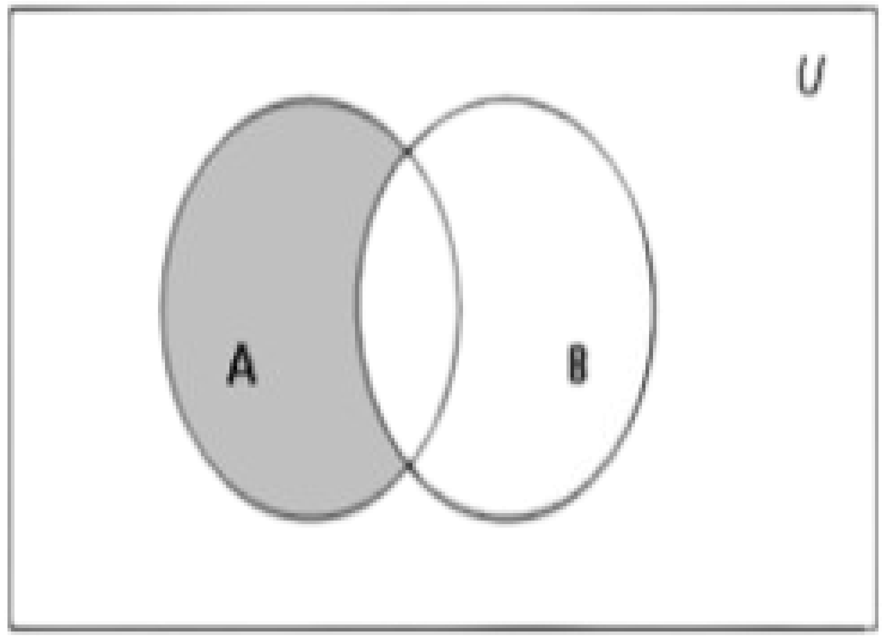
a) {0, 1, 2, 3, 4, 5, 6 }

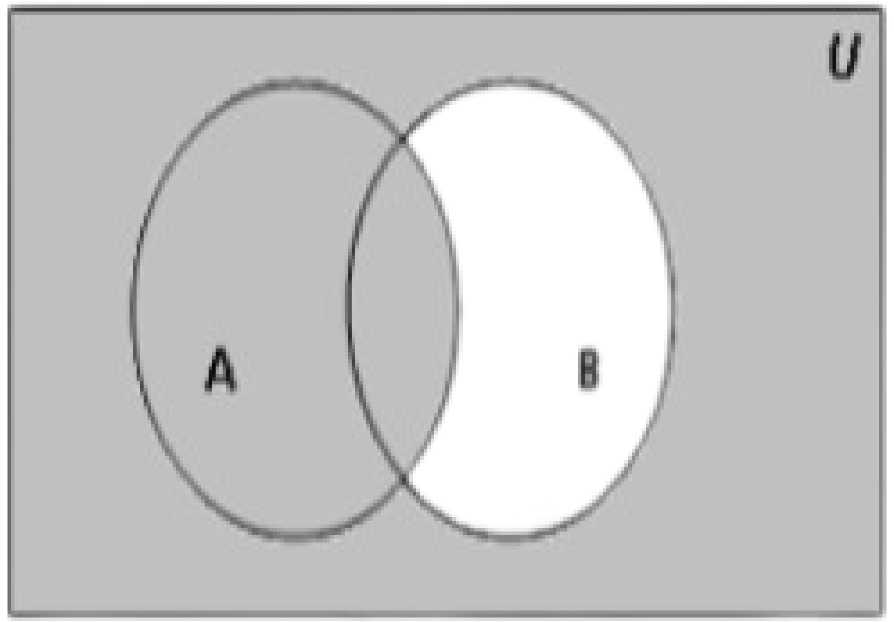
b) {3}

c) {1, 2, 4, 5}

d) {0, 6}

e) {0,1,2,4,5, 6}

2.a) 

b) 

EXAMPLE

List the members of these sets.

a) $\{x \mid a \text{ is real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Solution:

a) $\{-1 , 1\}$

b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

c) $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

d) Ø (empty set which means no element)

Exercise C

1. Given $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, $C = \{5, 6, 7, 8, 9\}$, $D = \{1, 3, 5, 7, 9\}$, $E = \{2, 4, 6, 8\}$, $F = \{1, 5, 9\}$.

   Find:

   a) $A \cup B$ e) $C \cup F$

   b) $B \cap D$ f) $E - B$

   c) $D \cap E$ g) $D \backslash A$

   d) $F \backslash A$ h) $\mathrm{B} \oplus C$

2. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find :

   a) $A \cup B$

   b) $A \cap B$

   c) $A - B$

   d) $B \oplus A$

EXERCISE C CONT…

3. Given $U = \{1,2,3,4,5,6,7,8,9\}, A = \{2,4,6,8\}, B = \{1,3,4,5,7\}, C = \{7,8\}$

a) $A \cap B$

$b)$ B'

$c)$ $A' \cup B'$

d) $n(A - B)$

$e)$ $(A - C)'$

f) $(A \cap C) \cap (A \cup B)$

4. Given $A = \{x: x \text{ is an integer and} -3 \leq x < 7)$ and $B = \{x: x \text{ is a natural number less than } 6)$

a) List all the elements of the set A and B

b) Find $A \oplus B$

RELATIONS

BINARY RELATION

- Definition: relationships between the elements of two sets.
- A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pairs comes from A and the second elements come from B.

IMPORTANT

Notation form:

*a ***R*** b (a is R-related to b)

*a $\not{R}$ b (a is not R-related to b)

Domain: all first elements of the ordered pairs which belong to ***R***

Range: the set of second elements in ***R***

EXAMPLE

Let A=(1,2,3) and B=(x,y,z) and let R={(1,y),(1,z),(3,y)}.Define the relation given in notation form. State the domain and range.

<u>Solution:</u>

1 ***R*** y, 1 ***R*** z, 3 ***R*** y

1 $\cancel{R}$ x, 2 $\cancel{R}$ x, 2 $\cancel{R}$ y

2 $\cancel{R}$ z, 3 $\cancel{R}$ x, 3 $\cancel{R}$ z

Domain={1,3}

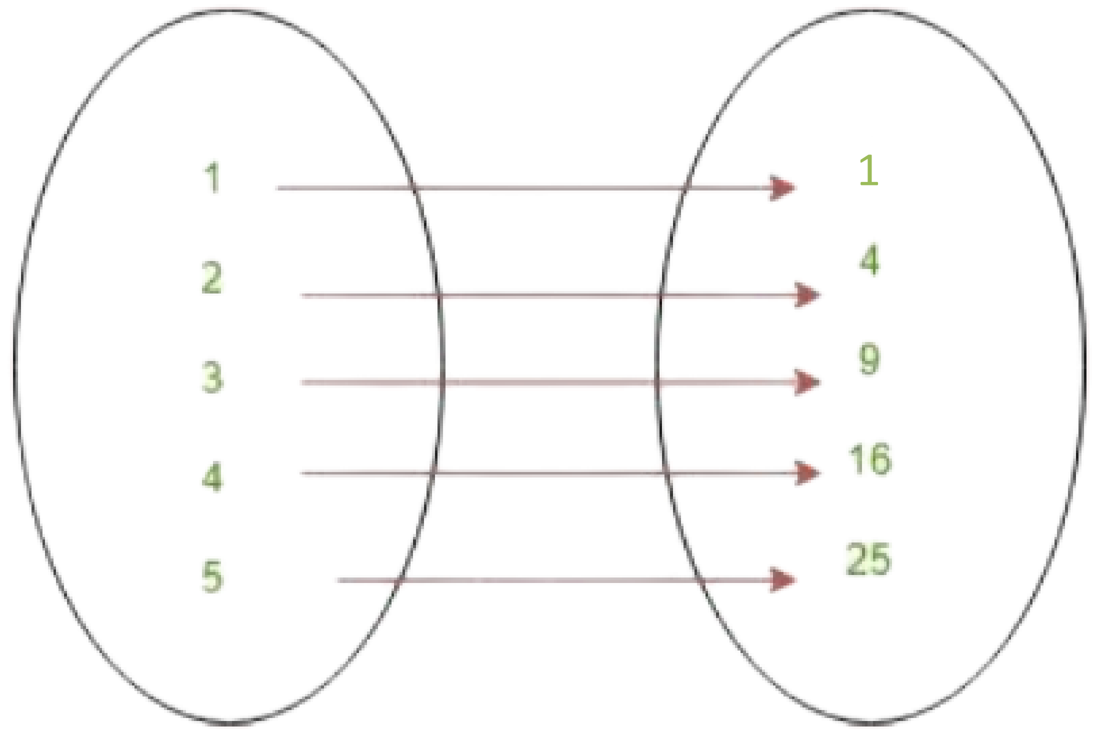
Range={y,z}

RELATION REPRESENTATION

1. GRAPHICALLY/ ARROW DIAGRAM

Example:

R= {(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)}



RELATION REPRESENTATION

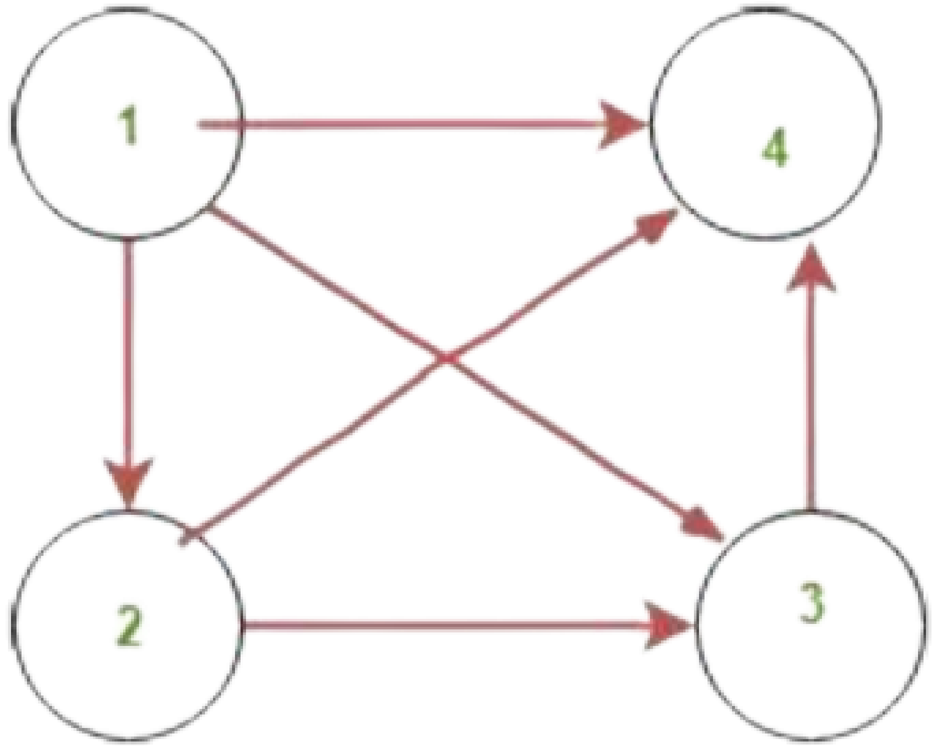
2. DIGRAPH (DIRECTED GRAPH)

- It consists of set 'V' of vertices and with the edges 'E'. Here E is represented by ordered pair of Vertices.
- In the edge (a, b), a is the initial vertex and b is the final vertex.
- If edge is (a, a) then this is regarded as loop.

Example:

R= {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}

Digraph:



RELATION REPRESENTATION

3. MATRICES

- The relation R is represented by the matrix $MR = [m_{ij}]$.
- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j

Example:

Suppose that A = {1,2,3} and B = {1,2}.

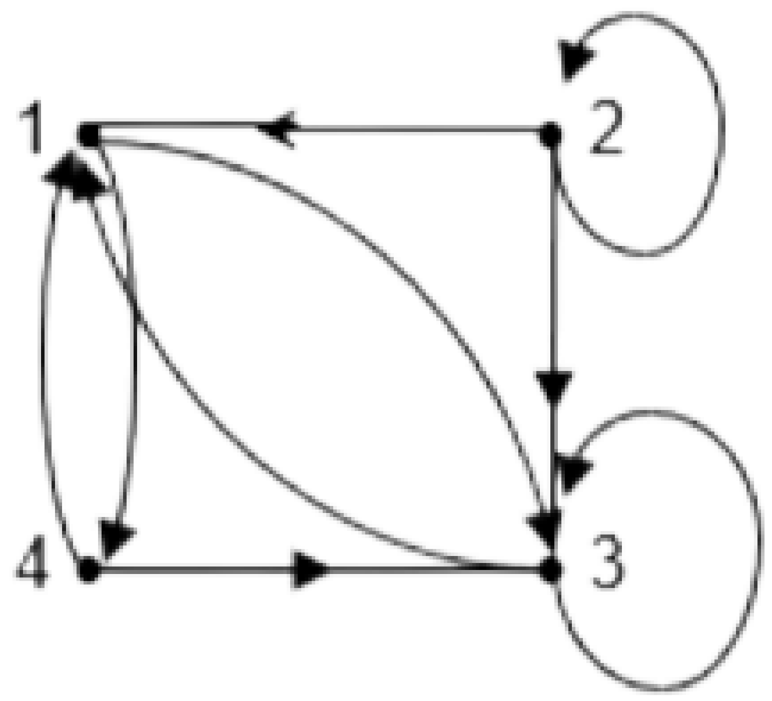
R= {(2, 1), (3,1), (3,2)}

In matrix form;

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise D

1. What are the ordered pairs in the relation R represented by the directed graph shown below?



2. Draw a directed graph (Digraph) for the relation
   a) R = { (1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)} on the set {1, 2, 3, 4}
   b) R = { (a,b), (a,a), (b,b), (b,c), (c,c), (c,b), (c,a)} on the set {a, b, c}

Exercise D cont…

3. **List** and display all the relation **graphically** the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3} where (a, b) ∈ R if and only if

   a) a = b
   b) a + b = 4
   c) a > b
   d) a divides b (a | b) *(*it means b / a)*

State the **domain and range** for all the questions above.

THE PROPERTIES OF RELATIONS IN DIRECTED GRAPH

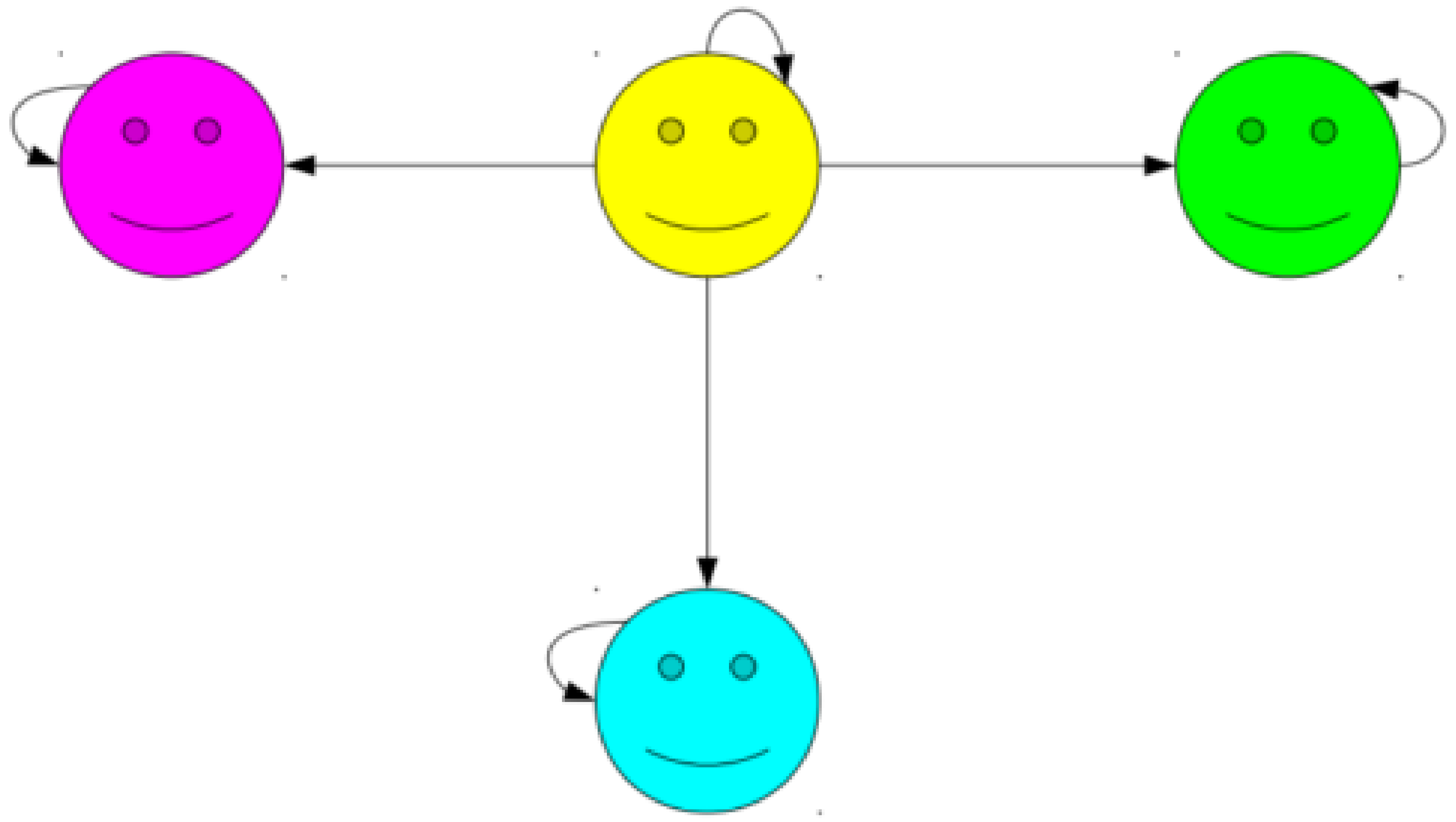
- A relation R is reflexive if there is loop at every node of directed graph.
- A relation R is irreflexive if there is no loop at any node of directed graphs.

- A relation R is symmetric if for every edge between distinct nodes, an edge is always present in opposite direction.
- A relation R is antisymmetric if there are never two edges in opposite direction between distinct nodes.

- A relation R is transitive if there is an edge from a to b and b to c, then there isalways an edge from a to c.

REFLEXIVE

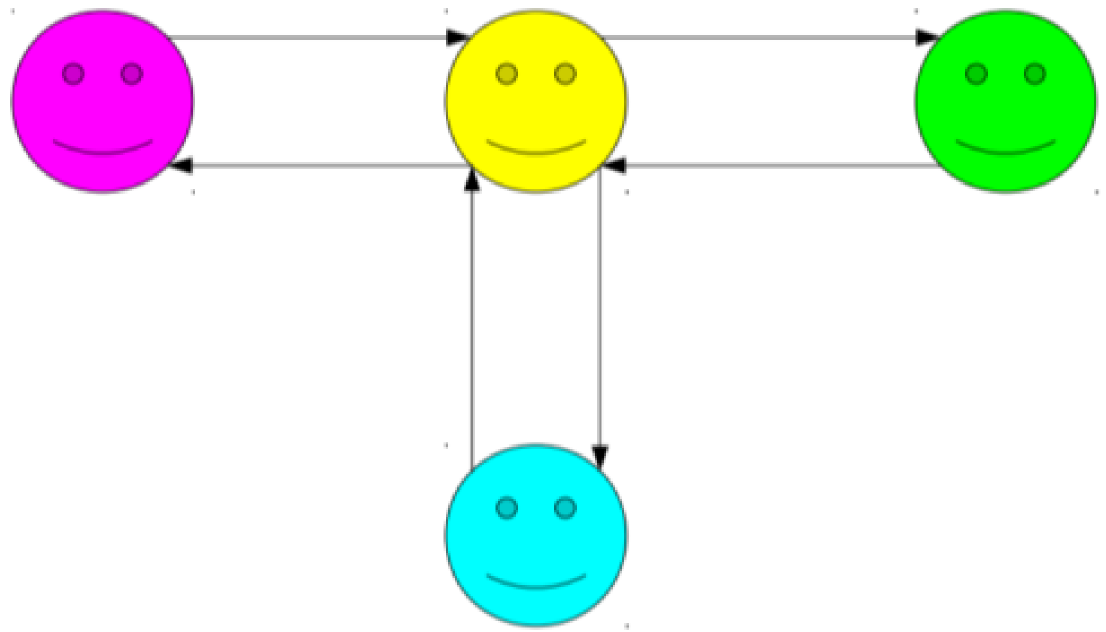
- A binary relation R over a set A is called reflexive iff For any $x \in A$, we have xRx.
- Look at the picture below;



- We have loop at every node of directed graph. Therefore, it is reflexive.

SYMMETRIC

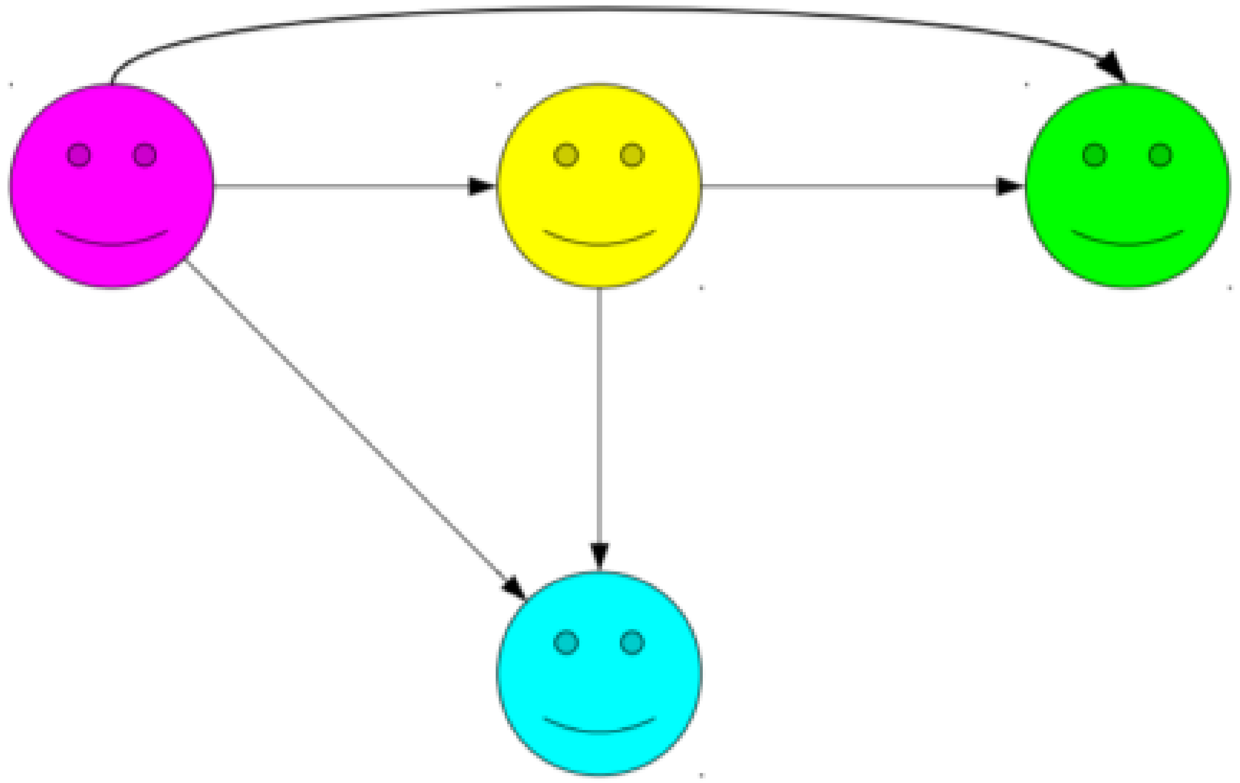
- A binary relation R over a set A is called symmetric iff For any $x \in A$ and $y \in A$, if xRy, then yRx.
- Look at the picture below;



- We have two edges with opposite direction for each nodes. Therefore, it is symmetric.

TRANSITIVE

- A binary relation R over a set A is called transitive iff For any x, y, z ∈ A, if xRy and yRz, then xRz.
- • Look at the picture below;



EXAMPLE

Consider the following relations on {1, 2, 3, 4} :

R_1 = {(1,1) , (1,2) , (2,1) , (2,2) , (3,4) , (4,1) , (4,4)} ;

R_2 = {(1,1) , (1,2) , (2,1)}

R_3 = {(1,1) , (1,2) , (1,4), (2,1) , (2,2) , (3,3) , (4,1) , (4,4)} ;

R_4 = {(2,1) , (3,1) , (3,2) , (4,1) , (4,2) , (4,3)}

R_5 = {(1,1) , (1,2) , (1,3) , (1,4) , (2,2) , (2,3) , (2,4) , (3,3) , (3,4) , (4,4)} ;

R_6 = {(3,4)}

Which of these relations are reflexive?

Solution:

The relations R_3 and R_5 are reflexive because they both contain all pairs of the form a, a , namely (1,1) , (2,2) , (3,3) , (4,4) .

EXAMPLE

Consider the following relations on 1, 2, 3, 4 :

$R_1 = \{(1,1)\ ,\ (1,2)\ ,\ (2,1)\ ,\ (2,2)\ ,\ (3,4)\ ,\ (4,1)\ ,\ (4,4)\}$;

$R_2 = \{(1,1)\ ,\ (1,2)\ ,\ (2,1)\}$

$R_3 = \{(1,1)\ ,\ (1,2)\ ,\ (1,4)\ ,\ (2,1)\ ,\ (2,2)\ ,\ (3,3)\ ,\ (4,1)\ ,\ (4,4)$;

$R_4 = \{(2,1)\ ,\ (3,1)\ ,\ (3,2)\ ,\ (4,1)\ ,\ (4,2)\ ,\ (4,3\ )$

$R_5 = \{(1,1)\ ,\ (1,2)\ ,\ (1,3)\ ,\ (1,4)\ ,\ (2,2)\ ,\ (2,3)\ ,\ (2,4)\ ,\ (3,3)\ ,\ (3,4)\ ,\ (4,4)\}$;

$R_6 = \{(3,4)\}$

Which of these relations are symmetric, not symmetric or antisymmetric?

<u>Solution:</u>

The relations R_2 and R_3 are symmetric. The relations R_1 are not symmetric

The relations R_4 , R_5 and R_6 are antisymmetric.

EXAMPLE

Consider the following relations on 1, 2, 3, 4 :

$R_1 = \{(1,1)\ ,(1,2)\ ,(2,1)\ ,(2,2)\ ,(3,4)\ ,(4,1)\ ,(4,4)\}$; $R_2 = \{(1,1)\ ,(1,2)\ ,(2,1)\}$

$R_3 = \{(1,1)\ ,(1,2)\ ,(1,4)\ ,(2,1)\ ,(2,2)\ ,(3,3)\ ,(4,1)\ ,(4,4)\}$;

$R_4 = \{(2,1)\ ,(3,1)\ ,(3,2)\ ,(4,1)\ ,(4,2)\ ,(4,3)$

$R_5 = \{(1,1)\ ,(1,2)\ ,(1,3)\ ,(1,4)\ ,(2,2)\ ,(2,3)\ ,(2,4)\ ,(3,3)\ ,(3,4)\ ,(4,4)\}$; $R_6 = \{3,4\}$

Which of these relations are transitive?

<u>Solution:</u>

The relations R_4 , R_5 and R_6 are transitive.

Not transitive:

R_1 because there’s (4,1) & (1,2) but there’s no (4,2)

R_2 because there’s (2,1) & (1,2) but there’s no (2,2)

R_3 because there’s (4,1) & (1,2) but there’s no (4,2)

Exercise E

1. Consider the relation (reflexive, symmetric and transitive) on the set { 1, 2, 3, 4, 5} : R = {(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)}, determine whether the relation R is reflexive, symmetric or transitive. Explain your answer.

2. Let M = {0, 1, 2,3} and defined relation R = {(0,1), (0,3), (1,0), (1,1), (2,3), (3,0), (3,2), (3,3)}

   a) Represent the relation R using directed graph

   b) Determine whether the relation R is reflexive, symmetric or transitive. Explain your answer.

Exercise E cont...

3. Given those three relations on set A = {1,2,3,4}:

$$R = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$S = \{(1,1),(1,3),(1,4),(3,4)\}$$
$$T = \{(1,2),(2,2),(2,3)\}$$

a) Determine which relations are reflexive. Give your reason.
b) Determine which relations are not symmetrical. Explain your answer.
c) Give a reason why S is transitive.

Exercise E cont...

4. Given A = {1,2,34}, B = {1,4,6,8,9} where element a is in A is related to element b in B, if and only if $b = a^2$.

   a) List the element of the relation
   b) Draw the directed graph for the relation
   c) Determine whether the relation is reflexive or not.
   d) Is the relation symmetric? Explain your answer.

EQUIVALENCE RELATIONS

A relation on a set A is called an *equivalence relation* if it is

- Reflexive
- symmetricand
- transitive.

Chapter 1: SETS, RELATIONS AND FUNCTION

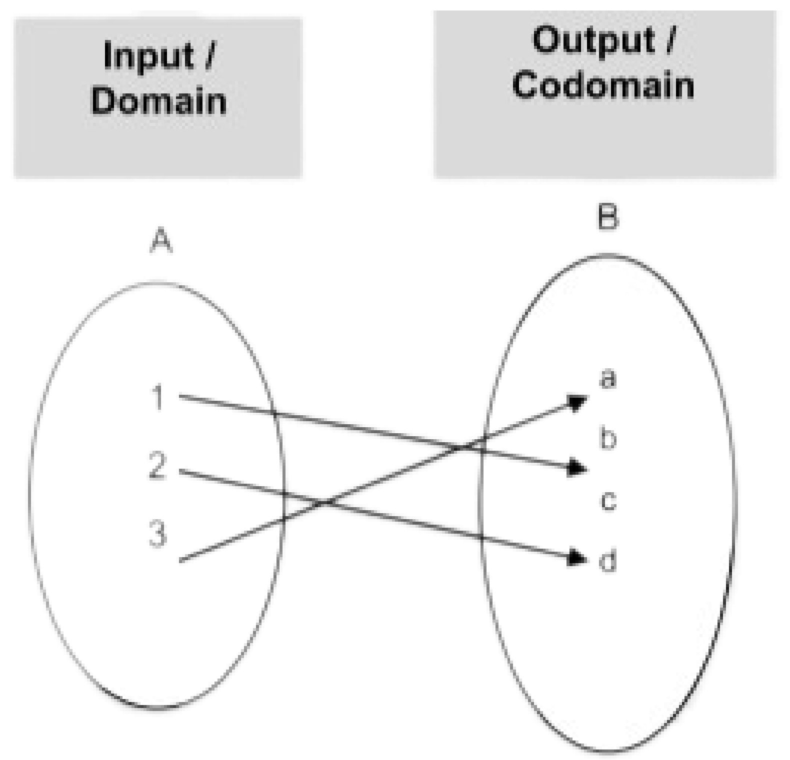
1.3 Carry out functions

- 1.3.1 Explain composite functions
- 1.3.2 Explain inverse functions

DEFINE FUNCTIONS

- **A function** is a relation that has exactly **one output for each possible input in the domain**.
- A function maps values to **one and only one value**.

Example:



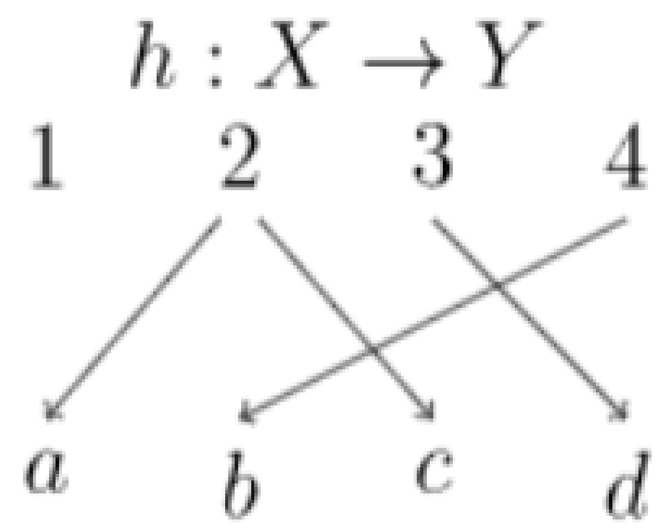
*All values in the domain has exactly one output only. Therefore, it is a function.

DEFINE FUNCTIONS

NOT A FUNCTION when:

- Any values in the domain has not been mapped to any element from the codomain.
- Any values in the domain have been mapped to more than one element from the codomain

<u>Example:</u>



- The element 1 from the domain has not been mapped to any element from the codomain.
- The element 2 from the domain has been mapped to more than one element from the codomain (a and c).

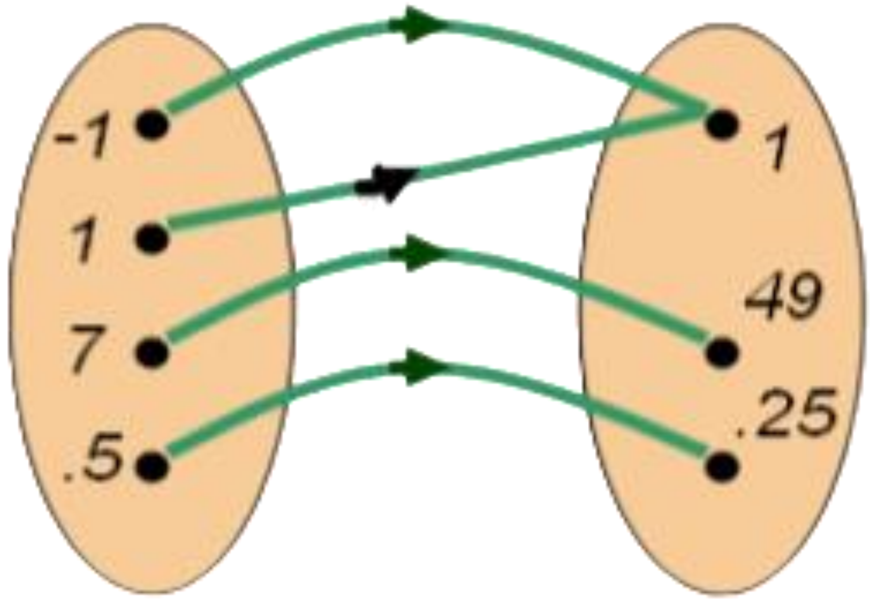
DESCRIBE BASIC CONSTRUCTIONS

<u>Example</u>:

If we write (define) a function as $f(x) = x^2$, then we say 'f of x equals x squared' and we have

$$f(-1) = 1;\ f(1) = 1;\ f(7) = 49;\ f(1/2) = 1/4;\ f(4) = 16 \text{ and so on.}$$

This function f maps numbers to their squares (as mapping/graphic below)

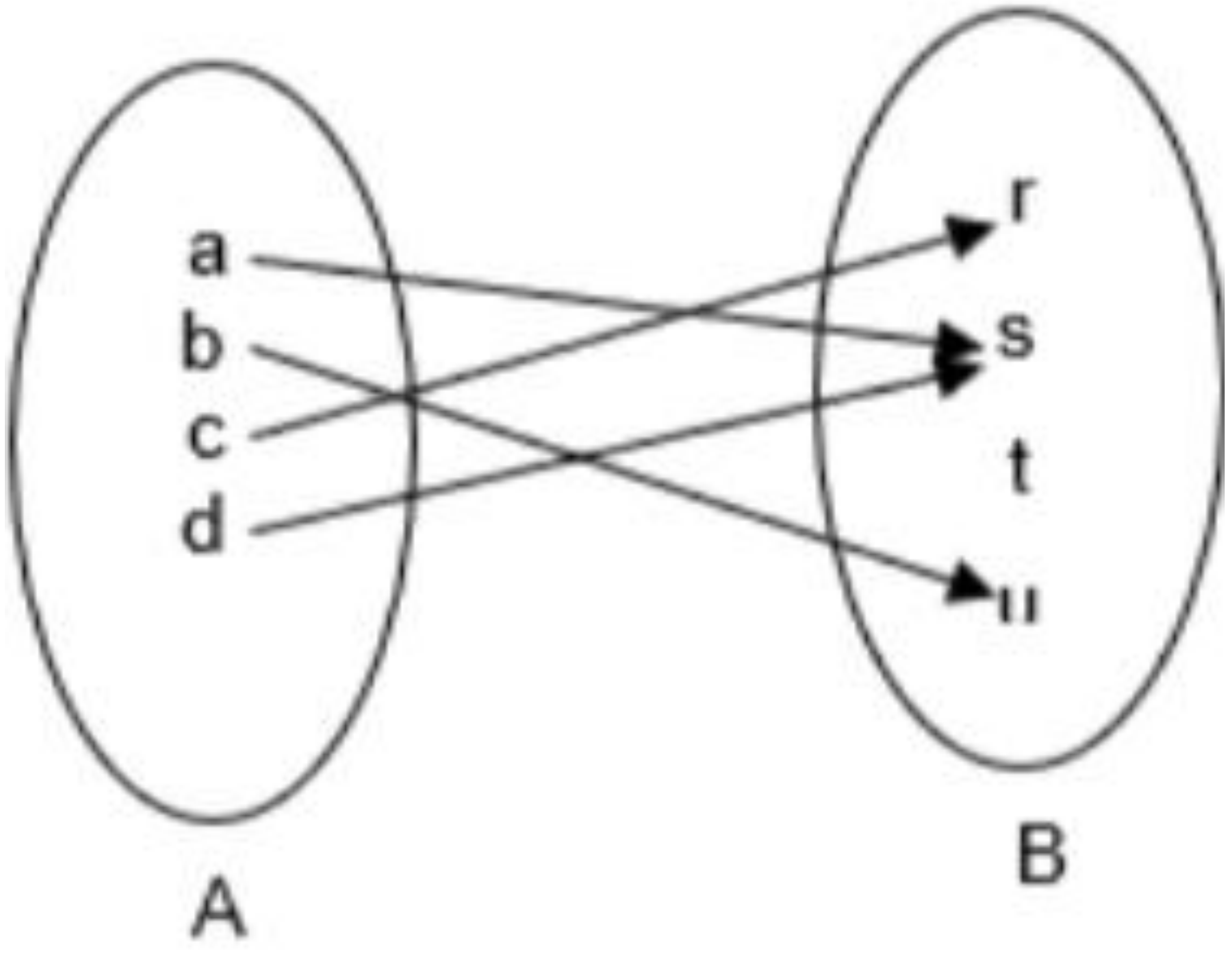


Remark: Functions are sometimes also called **mappings** or **transformations**

IMPORTANT TERMS USED IN FUNCTIONS

- The element in set A is called the **domain**
- The element in set B is called **codomain**
- Unique element of B which is assign to A is called **image / range**

EXAMPLE



Domain = {a, b, c, d}

Codomain = {r, s, t, u}

Range/Image = {r, s, u}

COMPOSITE FUNCTIONS

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by f o g, is defined by

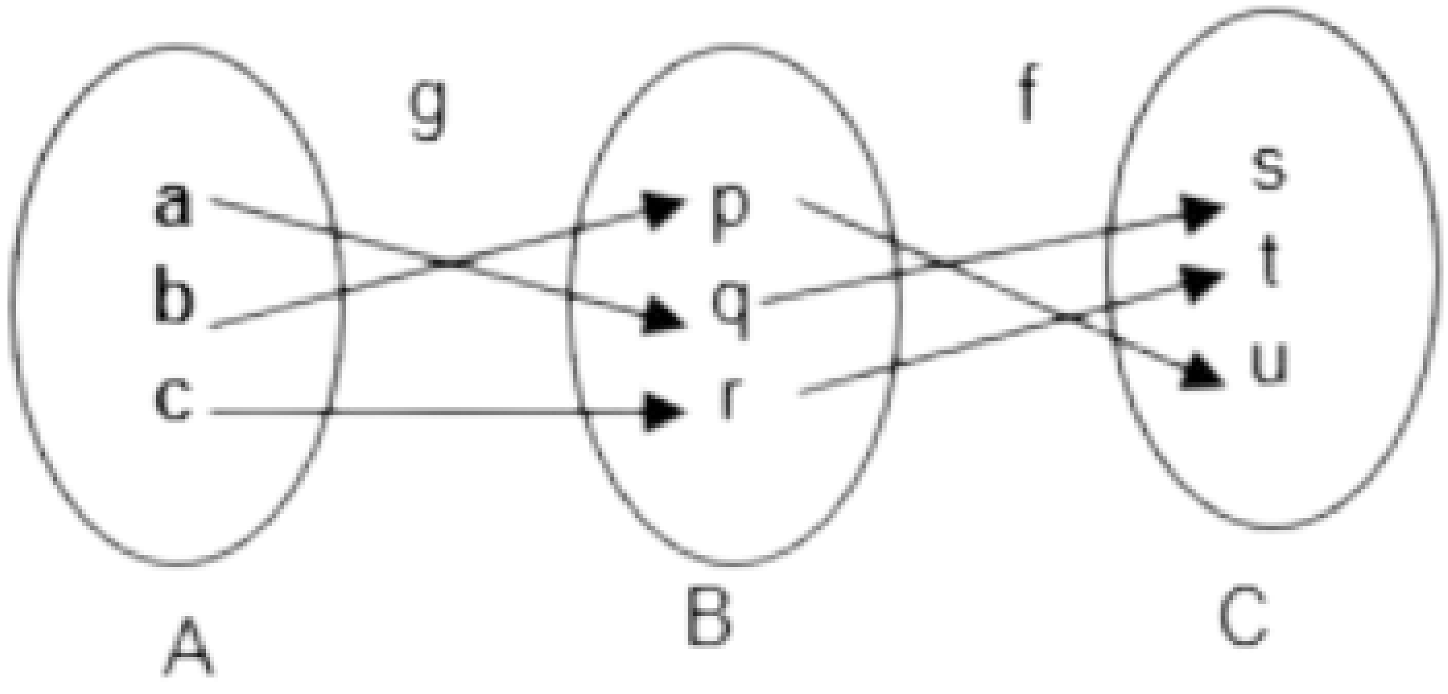
(f o g)(a) = f (g (a))

- In symbol: $g: A \rightarrow B, f: B \rightarrow C = f \circ g: A \rightarrow C$
- Composition function happened when codomain B is the domain in $f: B \rightarrow C$

EXAMPLE 1

Composition function $f \circ g$

$$g: A \rightarrow B, f: B \rightarrow C = f \circ g: A \rightarrow C$$



$$\therefore f \circ g: \{(a,s),(b,u),(c,t)\}$$

EXAMPLE 2

Let f and g be the functions from the set of integers to the set of integers defined by f(x) = 2x + 3 and g(x) = 3x + 2. What is $f \circ g$ and $g \circ f$?

Solution:

$f \circ g(x)$ = f (g(x)) = f (3x + 2)

= 2 (3x + 2) + 3

= 6x + 4 + 3

= 6x + 7

$g \circ f$ = g (f (x)) = g (2x + 3)

= 3 (2x + 3) + 2

= 6x + 9 + 2

= 6x + 11

EXAMPLE 3

Given f(x) = $\frac{3}{x}$ and fg(x) = 3x.

What is and g (x)?

<u>Solution:</u>

Given f (x) = $\frac{3}{x}$

fg (x) = $\frac{3}{g(x)}$ …...................(1)

Given fg (x) = 3x……….(2)

$$\frac{3}{g(x)} = 3x$$

$$\therefore g(x) = \frac{3}{3x}$$

$$= \frac{1}{x}$$

EXAMPLE 4

The function f is defined by $f(x) = x - 2$. Another function g is such that gf(x) = $\frac{1}{11-3x}$, x $\neq \frac{11}{3}$. Find the function g.

<u>Solution:</u>

Given $f\ (x) = x - 2$ and $gf(x) \ = \frac{1}{11-3x}$

gf (x) = $\frac{1}{11-3x}$

g[f(x)] = $\frac{1}{11-3x}$

g[$(x - 2)$] = $\frac{1}{11-3x}$

Let $x - 2 = u, then\ x = u + 2$

$$\therefore g(u) = \frac{1}{11-3(u+2)}$$

$$= \frac{1}{11-3u-6}$$

$$= \frac{1}{5-3u}$$

$$\therefore g(x) = \frac{1}{5-3x}, \text{x} \neq \frac{5}{3}$$

Exercise F

1. Given that $f(x) = 3x + x^2$, $g(x) = \frac{x}{2} - 4$ and $h(x) = x - 5$.

Determine,

a) $\mathrm{fh(x)}$
b) $\mathrm{gh(2)}$
c) $\mathrm{h^2(6)}$

2. Given $f(x) = 4x - 1$ and $g(x) = x^2 + 3$, find

a)$\mathrm{fg(-3)}$
b)The value of x when $\mathrm{f^2(x) = 7}$

Exercise F

3. Given that $f(x)= 2x+x^2$ and $g(x)= 1 - \frac{x}{4}$

a) $f(3)$

b) $fg(-4)$

4. Let f and g be functions from the positive integers to the positive integers defined by equations $f(n) = 2n + 1$, $g(n) = 3n - 1$. Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

Exercise F

5. The function *f* and the composite function *gf* are defined as follows. Find the function *g*.

a) $f(x) = x + 1, gf(x) = \frac{3}{x-2}, x \neq 2$

b) $f(x) = 3x + 2, gf(x) = 9x^2 + 9x + 2$

6. Given the functions $f(x) = 3x + 7$ and $fg(x) = 22 - 3x$, find $gf(x)$.

INVERSE FUNCTION

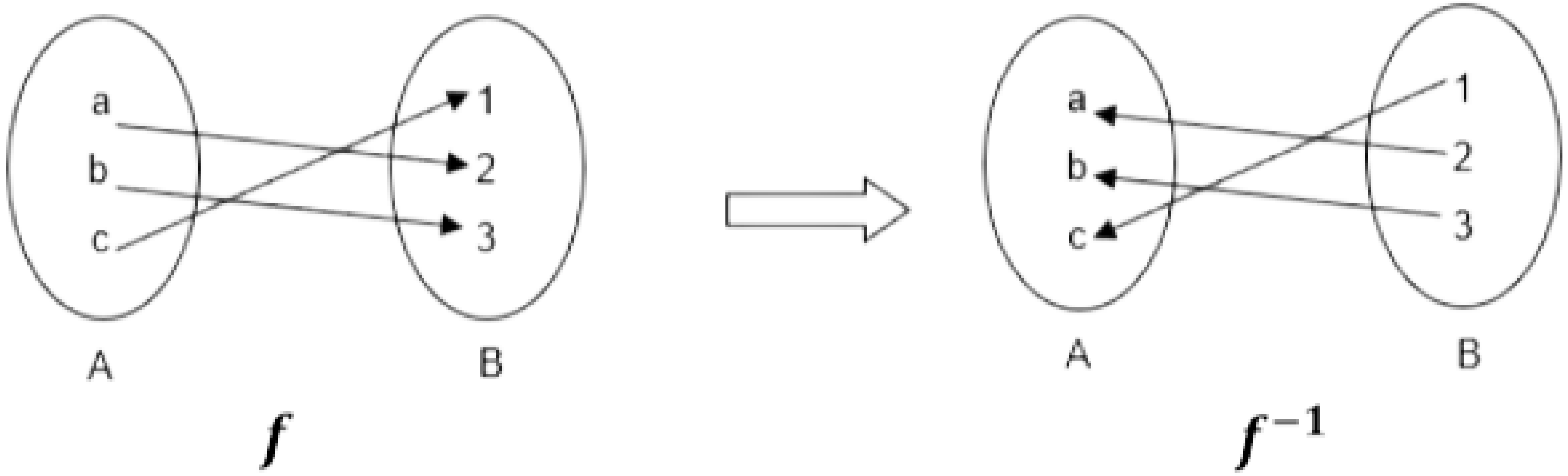
- If a function f from A to B is a one-to-one, then there is a function from B to A that "undoes" f, that is: it sends each element of B back to the element of A that it came from via f.
- The function is invertible if a function f is Onto function and at the same time it also one-to-one function.
- The **inverse function of f is denoted: f^{-1}.**

EXAMPLE

Let f be the function from {a, b, c} to {1, 2, 3} defined by f(a) = 2, f(b) = 3, and f(c) = 1. Is f invertible, and if it is, what is its inverse?

Solution:

The function f is invertible because it is a one-to-one. The inverse function f^{-1} reverse the correspondence given by f, so $f^{-1}(2) = a$, $f^{-1}(3) = b$, and $f^{-1}(1) = c$. This illustrated in figure below.



EXAMPLE

Given the function $f(x) = \frac{x+8}{x-6}$, $x \neq 6$. Find the value $\boldsymbol{f}^{-1}(3)$.

<u>Solution:</u>

Let $y = f^{-1}(3)$

$f(y) = 3$

Substitute the coefficient x with y into f(x)

$\frac{y+8}{y-6} = 3$

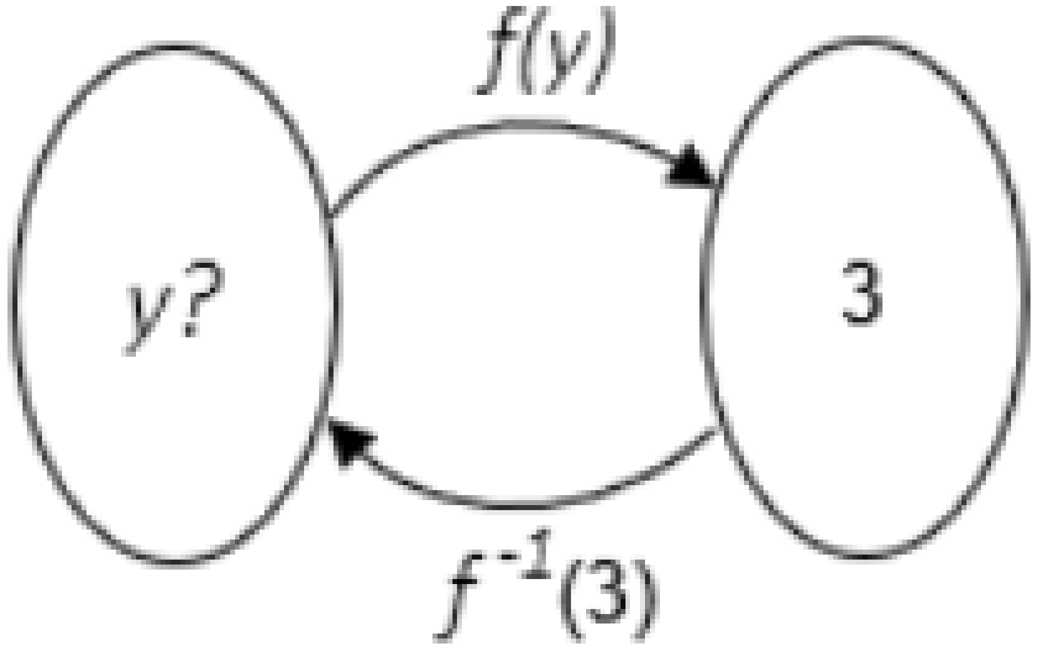
$y + 8 = 3(y - 6)$; $y + 8 = 3y - 18$

Find the value of y;

$3y - y = 8 + 18$

$2y = 26$; $y = 13$

$y = f^{-1}(3) \rightarrow f^{-1}(3) = 13$



EXAMPLE

Given f(x) = 2 + x^2 and $g \circ f$(x) = $\frac{1}{x^2}$

Find g(x) ?

<u>Solution:</u>

Step 1, find f^{-1}(x)

Let f^{-1}(x) =y

x = f(y)

x = 2 + y^2

y^2= x-2

y = √(x−2)

∴ $\boldsymbol{f^{-1}(x)}$ = √(x−2)

Step 2, substitute f^{-1} (x) into gf(x)

∴ $gf(\boldsymbol{f^{-1}(x)}) = \frac{1}{(\sqrt{x-2})^2}$

$g(x) = \frac{1}{x-2}$

TIPS:

1- Find $\boldsymbol{f^{-1}x}$

2-Substitute $\boldsymbol{f^{-1}x}$ into *gf(x)*

Exercise G

1. Given g = {(1,b), (2,c), (3,a)}, a function from X = {1, 2, 3} to Y = {a, b, c, d} and f = {(a,x), (b,x), (c,z), (d,w)}, a function from Y to Z = {w, x, y, z}, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.

2. Given that the function f(x) = 4x + 1, find a formula for $f^{-1}(x)$.

Exercise g cont…

3. Given that the functions $g(x) = x - 2$ and $f(g(x)) = x^2 - 4x + 8$. Find:

   a) $g(8)$

   b) The values of x if $f(g(x)) = 13$

   c) The function f

   d) $g^{-1}(-1)$

Exercise g cont…

4. Let $f(x) = 2x + 1$ and $g(x) = 3x^2 - 4$. Find:

   a) $(f \circ g)(x)$

   b) $(g \circ f)(-2)$

   d) $f^{-1}(x)$

5. The function f and g are defined by $f : \longmapsto 3x, g(x) \mapsto x + 2$

   a) Find an expression for $(g \circ f)(x)$

   b) Show that $f^{-1}(18) + g^{-1}(18) = 22$

Exercise g cont…

6. The information below defines the functions h and g

$$h : x \to 2x - 3$$
$$g : x \to 4x - 1$$

Find $gh^{-1}(x)$